Extra slides for Chapter 3: Adequacy of connectives

Based on Prof. Lila Kari’s slides
For CS2209A/B
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A remarkable property of the standard set of connectives \((\sim, \bullet, \lor, \supset, \equiv)\) is the fact that for every table

\[
\begin{array}{|c|c|c|}
\hline
P & Q & \ldots \\
\hline
1 & 1 & \ldots \\
1 & 0 & \ldots \\
0 & 0 & \ldots \\
\hline
\end{array}
\]

there is a formula (depending on the variables \(P, Q, \ldots\) and using only the standard connectives) that has exactly this truth table.

(There is Boolean function \(f(P, Q, \ldots)\) with exactly this truth table).

Any set of connectives with the capability to express all truth tables is said to be adequate. As Post (1921) observed, the standard connectives are adequate.

We can show that a set \(S\) of connectives is adequate if we can express all the standard connectives in terms of \(S\).
Adequate set of connectives

Formulas $(A \supset B)$ and $(\neg A \lor B)$ are tautologically equivalent. Then $\supset$ is *definable* in terms of (or is reducible to or can be expressed in terms of) $\neg$ and $\lor$.

Similarly, $\lor$ is definable in terms of $\neg$ and $\supset$ because $(A \lor B)$ is tautologically equivalent to $(\neg A \supset B)$

**Theorem.** $\{\neg, \bullet, \lor\}$ is an adequate set of connectives (given the standard set of connectives is adequate)

**Proof.**
For any formulas $A, B$

$$ (A \supset B) = (\neg A \lor B) $$

and

$$ (A \equiv B) = ((A \supset B) \bullet (B \supset A)) $$

**Corollary** $\{\neg, \bullet\}, \{\neg, \lor\}$ and $\{\neg, \supset\}$ are adequate, given....

**Proof.** Exercise.
Proving inadequacy

How do we show that a given set of connectives is not adequate? Show that some standard connective cannot be expressed by $S$.

**Example.** The set $S = \{ \bullet \}$ is not adequate.

**Proof.** To see this, note that a formula depending on only one variable and which uses only the connective $\bullet$ has the property that its truth value for a value assignment that makes $P = 0$ is always 0.

In order to define the negation $\neg P$ in terms of $\bullet$, there should exist a formula $F$ depending on the variable $P$ and using only the connective $\bullet$ such that $\neg P = F$.

However, for a value assignment $\nu$ such that $\nu(\neg P) = 1$, we have $\nu(P) = 0$ and therefore $\nu(F) = 0$, which shows that $\neg P$ and $F$ cannot be tautologically equivalent.
Adequate set of connectives

Schroder showed in 1880 that each of the standard connectives is definable in terms of a single binary connective $\downarrow$, where the truth table associated with $\downarrow$ is

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>$P \downarrow Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</tbody>
</table>

We can express $\downarrow$ in terms of the standard connectives by $(P \downarrow Q) = (\neg P \cdot \neg Q)$, and also the standard connectives in terms of $\downarrow$ by

\[
\begin{align*}
\neg P &= (P \downarrow P) \\
(P \cdot Q) &= (P \downarrow P) \downarrow (Q \downarrow Q) \\
(P \lor Q) &= (P \downarrow Q) \downarrow (P \downarrow Q) \\
(P \Rightarrow Q) &= ((P \downarrow P) \downarrow Q) \downarrow ((P \downarrow P) \downarrow Q) \\
(P \equiv Q) &= ((P \downarrow P) \downarrow Q) \downarrow ((Q \downarrow P) \downarrow P)
\end{align*}
\]

Thus it follows that a single connective $\downarrow$ is adequate. Consequently, to test a given S for being adequate it suffices to test if $\downarrow$ can be expressed by S.
Adequate set of connectives

In 1913 Sheer showed that the Sheer stroke | with associated truth table

<table>
<thead>
<tr>
<th>P</th>
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<th>P</th>
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</thead>
<tbody>
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is also a single binary connective in terms of which the standard connectives can be expressed. Prove it is adequate given …